Ergodic Theory - Week 5

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1 Classifying measure preserving systems

P1. Prove that the systems $\mathbb{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha)$ and $\mathbb{Y} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2)$ are not isomorphic.

Arguing by contradiction, assume there exists a factor map $\phi : (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha) \to (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2)$ and a factor map $\psi : (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2) \to (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha)$ such that $\psi(\phi(x)) = x$ and $\phi(\psi(x))$ for almost all $x \in \mathbb{T}$. This means that ϕ is invertible (with $\phi^{-1} = \psi$) in a full-measure subset X of $\mathbb{T}$. Thus $T_2 = \phi \circ R_\alpha \circ \phi^{-1}$ on X which implies T_2 is invertible almost everywhere on X. Let A be the set on which T_2 is invertible, so that $\mu_{\mathbb{T}}(A) = 1$. Take $x \in A$ and observe that

$$T_2\Big(x + \frac{1}{2}\Big) = 2x + 1 \pmod 1 = 2x \pmod 1 = T_2(x),$$

thus $x + \frac{1}{2} \notin A$. Therefore $m_{\mathbb{T}}(A) \leq \frac{1}{2}$ which is a contradiction.

P2. Let $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$. Show that there is an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$||n_k \alpha_l||_{\mathbb{T}} \leq \frac{1}{k}, \quad \forall l \in \{1, \dots, k\}.$$

Hint: Use Poincaré's Theorem in a convenient group rotation.

Define $n_0 \in \mathbb{N}$ arbitrarily. Assume that we have defined $n_{k-1} \in \mathbb{N}$ satisfying the desired property. Consider the group rotation $(\mathbb{T}^k, \mathcal{B}(\mathbb{T}^k), \mu, R)$ in where μ is the Haar measure of $\mathbb{T}^k$ (in this case, the projection of the Lebesgue measure from $\mathbb{R}^k$) and R is the rotation by the element $(\alpha_1, \dots, \alpha_k)$. Thus,

$$T(x_1, \dots, x_k) = (x_1 + a_1 \pmod 1, \dots, x_k + a_k \pmod 1).$$

The distance in $\mathbb{T}^k$ given by $d(x, y) = \text{máx}_{i\in[k]} ||x_i - y_i||_{\mathbb{T}}$ generates the topology of $\mathbb{T}^k$, where for $t \in \mathbb{T}$ we denote $||t||_{\mathbb{T}} = \text{mín}(1 - t, t)$ which is the distance to the nearest integer.

Set $U = B(0, \frac{1}{2k})$, which has positive measure. Then by Poincaré's Theorem, there exists $n_k > n_{k-1}$ such that

$$\mu(U \cap R^{-n_k} U) > 0.$$

This implies that there exists $x \in U$ such that $R_{n_k} x \in U$. Therefore, we have that

$$||(n_k\alpha_1, \dots, n_k\alpha_k)||_{\mathbb{T}^k} = ||R_{n_k} x - x||_{\mathbb{T}^k} < \frac{1}{k},$$

which implies

$$||n_k\alpha_l||_{\mathbb{T}} \leq \frac{1}{k}, \quad \forall l \in \{1, \dots, k\}.$$

P3. Prove that the systems $\mathbb{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_4)$ and $\mathbb{Y} = (\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), m_{\mathbb{T}^2}, T_2 \times T_2)$ are isomorphic.

First of all, we define $\mathbb{N}[\frac{1}{2}] = \{\frac{m}{2^n} : m, n \in \mathbb{N}\}$ and let $X = [0, 1) \setminus \mathbb{N}[\frac{1}{2}]$. Notice that $m_{\mathbb{T}}(X) = 1$ and every element in X has unique expansion in base 2. We define the map $\Phi : X \to X^2$ given by

$$\Phi(\sum_{n=1}^{\infty} \frac{x_n}{2^n}) = \Big(\sum_{n=1}^{\infty} \frac{x_{2n-1}}{2^n}, \sum_{n=1}^{\infty} \frac{x_{2n}}{2^n}\Big),$$

for $x = 0.x_1x_2x_3...$ in base 2. Then ϕ is a well defined bijection between X and X^2, where the latter has full measure in $\mathbb{T}^2$.

Firstly, we prove that ϕ preserves the measure. For this, we identify (isomorphically) $\mathbb{T}$ with $\{0, 1\}^{\mathbb{N}_0}$. Then we consider two cylinders A and B in $\{0, 1\}^{\mathbb{N}_0}$, being the sets of $0.x_1x_2x_3\dots$ with the first k and m coordinates fixed. Hence, $\phi^{-1}(A \times B)$ has $k + m$ coordinates fixed. Therefore

$$m_{\mathbb{T}}(\phi^{-1}(A \times B)) = 2^{-(k+m)} = 2^{-k}2^{-m} = m_{\mathbb{T}^2}(A)m_{\mathbb{T}^2}(B) = m_{\mathbb{T}^2}(A \times B).$$

Finally, we show that $\phi \circ T_4 = (T_2 \times T_2) \circ \phi$. For $x = 0.x_1x_2x_3.... \in X$ we have

$$\phi \circ T_4(x) = \phi(0.x_1x_2x_3... \pmod 1) = \phi(x_1x_2.x_3x_4\dots \pmod 1) = \phi(0.x_2x_3...) = (0.x_2x_4..., 0.x_3x_5...),$$

and

$$\begin{aligned}(T_2 \times T_2) \circ \Phi(x) = (T_2(0.x_0x_2...), T_2(0.x_1x_3...)) &= (x_0.x_2x_4 \pmod 1, x_1.x_3x_5... \pmod 1)\\ &= (0.x_2x_4..., 0.x_3x_5...)\end{aligned}$$

and we reach the desired conclusion.

P4. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system and let A be a set of positive measure. The pointwise ergodic theorem shows that for almost all $x \in X$, the set of visiting times

$$\Lambda_x = \{n \in \mathbb{N} \colon T^n x \in A\}$$

has natural density equal to $\mu(A)$. Is it true that, for almost all $x \in X$, the set Λ_x has bounded gaps?

The answer is negative. We will actually give an example where, for almost all x, the set Λ_x does not have bounded gaps.

Let $(\mathbb{T}, \mathcal{B}(T), \mu)$ be the circle with the Borel σ-algebra and the Lebesgue measure and consider the map $Tx = 10x \pmod 1$. Consider the set $A = [0.5.0.6]$ which has positive measure

We know that almost all $x \in \mathbb{T}$ are normal in base 10 and, thus, attain all patterns of digits infinitely often. Take a normal number x, let $a_n(x)$ denote its n-th digit in base 10 and assume that $\Lambda_x = \{n \in \mathbb{N} \colon \{10^n x\} \in A\}$ has gaps bounded by K (we allow K to depend on x). Consider the pattern $(1, \dots, 1)$ consisting of $2K$ consecutive 1s. Then, we know from the definition of normality that we have $a_n(x) = a_{n+1}(x) = \dots = a_{n+2K-1}(x) = 1$ for infinitely many n. Therefore, $\{10^m x\} \leq 0.2$ for all $m \in \{n, n + 1, \dots n + 2k - 1\}$. In particular,

$\{10^m x\} \notin A$ for all m in this range, contradicting the fact that Λ_x has gaps bounded by K.